WELCOME

Stochastic Models via Pathway Idea

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Objective

- In data analysis, the first step is to build an appropriate mathematical or stochastic model to describe the data so that further studies can be done with the help of the models.
- Here we consider several types of situations and the appropriate models to describe each situation.
- Input-output type mechanism is considered first, where reactions, diffusions, reaction-diffusion, production-destruction, decay type physical situations can fit in.
- Techniques are described to make thicker or thinner tails in stochastic models.
- Pathway idea is described where one can switch around to different functional forms through a parameter called the pathway parameter.

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Introduction

- Here we consider models which describe short-term behavior of data or behavior within one cycle if a cyclic behavior is noted.
- For example, when monitoring solar neutrinos it is seen that there is likely to be an eleven year cycle and within each cycle the behavior of the graph is something like slow increase with several local peaks to a maximum peak and slow decrease with humps back to normal level.
- Similarly, while monitoring the production of the chemical called melatonin in human body the nightly cycle each night shows the same type of behavior of slow rise starting in the evening with local peaks to a maximum peak, then slow decrease, with humps, back to normal level by the morning.
- In such situations, what is observed is not really what is actually produced.
- Many of natural phenomena belong to this type of behavior of the form $u = x - y$ where x is the input or production variable and y is the output or consumption or destruction or decay variable and u represents the residual part which is observed.
- A general analysis of input-output situation may be seen from Mathai (1993).

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- In reaction-rate theory, certain particle may react with each other in short-span or short-time periods and produce small number of particles, others may take medium time internals and produce larger numbers of particles and yet others may react over a long span and produce larger number of particles.
- For describing such types of situations in the production of solar neutrinos we consider mathematical models by erecting triangles whose ares are proportional to the neutrinos produced, see Haubold and Mathai (1995).
- Another approach that was adopted was to assume x and y as independently distributed random variables, then work out the density of the residual variable under the assumption that $x - y \geq 0$.
- The simplest such situation is an exponential type input and an exponential type output.
- Then the input-output model has the Laplace density, when x and y are identically and independently distributed and the density is given by,

$$f_1(u) = \frac{\beta_1}{2} e^{-\beta_1 |u-\alpha_1|}, 0 \leq u < \infty, \beta_1 > 0 \tag{1.1}$$

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- Suppose that this situation is repeated at successive locations and with the scale parameter $\beta = \beta_1, \beta_2, \ldots$. Then the nature of the graph will be that of a sum of Laplace densities. If the location parameters are sufficiently farther apart then the behavior of the graph is shown in the following Figure 1(a):

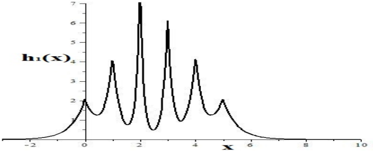
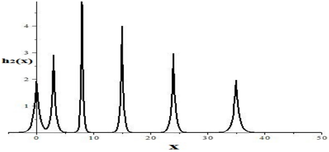


Figure 1(a)

- If such blips are occurring sufficiently close together then we have a graph of the type in Figure 1(b).



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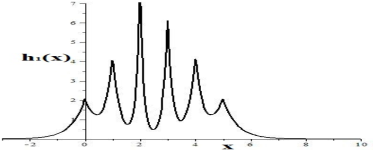

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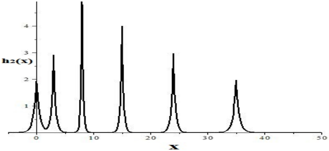

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- Here β_1 measures the intensity of the blip and α_1 the location where it happens, and each blip is the residual effect of an exponential type input and an independent exponential type output of the same strength.
- If $\alpha_1, \alpha_2, \ldots$ are farther apart then the contributions coming from other blips will be negligible and if $\alpha_1, \alpha_2, \ldots$ are close together then there will be contributions from other blips.
- Then the function will be of the following form:

$$h(u) = \sum_{j=1}^{k} \frac{\beta_j}{2} e^{-\beta_j |u-\alpha_j|}, 0 \le u < \infty, \beta_j > 0, j = 1, \ldots, k < \infty \qquad (1.2)$$

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- A symmetric Laplace density will be of the following form:

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and the graph is of the following form:

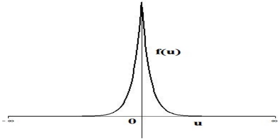

Figure 2: Symmetric Laplace density

- If the behavior of u is different for $u < 0$ and $u \geq 0$ then we get the asymmetric Laplace case which can be written as

$$g(u) = \begin{cases} \frac{1}{(\beta_1+\beta_2)} e^{\frac{u}{\beta_1}}, & -\infty < u < 0 \\ \frac{1}{(\beta_1+\beta_2)} e^{-\frac{u}{\beta_2}}, & 0 \leq u < \infty \end{cases} \qquad (1.4)$$

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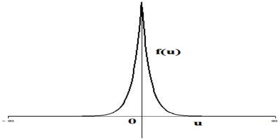

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- The graph of $g(u)$ is given in the following figure:

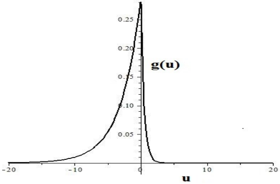

Figure 3: Asymmetric Laplace case

- When $\alpha_1 > 1, \alpha_2 > 1, \alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta$ we have independently and identically distributed gamma random variables for x and y and $u = x - y$ is the difference between them. Then $g_1(u)$ can be seen to be the following:

$$g_1(u) = \frac{u^{2\alpha-1}e^{-\frac{u}{\beta}}}{\beta^{2\alpha}\Gamma^2(\alpha)} \int_{z=0}^{\infty} (1+z)^{\alpha-1} z^{\alpha-1} e^{-\frac{1}{\beta}(2uz)} dz \qquad (1.5)$$

for $u \geq 0, \alpha > 0, \beta > 0$. This behaves like a gamma density and provides a symmetric model for $u \geq 0$ and $u < 0$.

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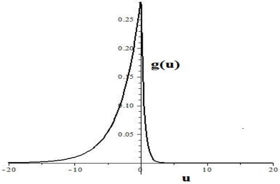

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- Graph is given in the following Figure 4:

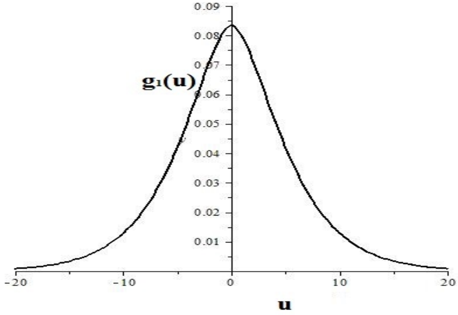

Figure 4: $g_1(u)$ in the symmetric gamma type input-output variables

Gamma model with appended Mittag-Leffler function

- Consider a gamma density of the type

$$g_3(x) = c_1\, x^{\gamma-1} e^{-\frac{x}{\delta}},\ \delta > 0,\ \gamma > 0,\ x \geq 0.$$

- Suppose that we append this $g_3(x)$ with Mittag-Leffler function $E^{\beta}_{\alpha,\gamma}(-x^{\alpha})$ where

$$E^{\beta}_{\alpha,\gamma}(-ax^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}(-a)^k \frac{x^{\alpha k}}{\Gamma(\gamma+\alpha k)},\ \alpha > 0,\ \gamma > 0.$$

- Then we get the density as

$$f^*(x) = \frac{(1+a\delta^{\alpha})^{\beta}}{\delta^{\gamma}} x^{\gamma-1} e^{-\frac{x}{\delta}} \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} \frac{(-a)^k \delta^{\alpha k}}{\Gamma(\gamma+\alpha k)}$$

for $0 \leq x < \infty, \alpha > 0, \gamma > 0, \delta > 0, \beta > 0, |a\delta^{\alpha}| < 1, a\beta\delta^{\alpha} < 1$.

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Gamma model with appended Mittag-Leffler function

- The following are some graphs of the appended Mittag-Leffler-gamma density. When $a < 0$ we have thinner tail and when $a > 0$ we have thicker tails compared to the gamma tail.

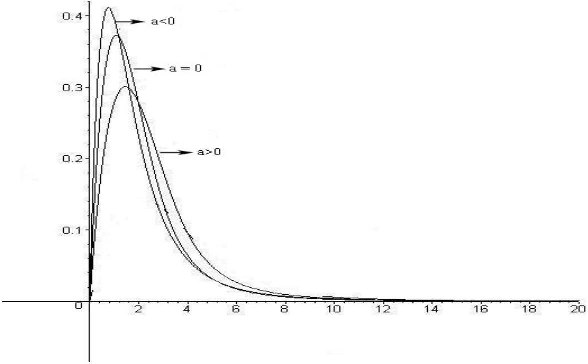

Figure 5: Gamma density with Mittag-Leffler function appended

Bessel appended gamma density

- Consider the model of the type of a basic gamma density appended with a Bessel function

$$\tilde{f}(x) = c\, x^{\gamma-1} e^{-\frac{x}{\delta}} \sum_{k=0}^{\infty} \frac{x^k(-a)^k}{k!\Gamma(\gamma+k)},\ \delta > 0, \gamma > 0, x \geq 0$$

where c is the normalizing constant.

- The appended function is of the form

$$\frac{1}{\Gamma(\gamma)}\,{}_0F_1(\ ;\gamma : -ax)$$

which is a Bessel function.

- Hence the density is of the form

$$\tilde{f}(x) = \frac{e^{a\delta}}{\delta^{\gamma}} x^{\gamma-1} e^{-\frac{x}{\delta}} \sum_{k=0}^{\infty} \frac{x^k(-a)^k}{k!\Gamma(\gamma+k)}, x \geq 0,\ \gamma > 0, \delta > 0.$$

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Bessel appended gamma density

- The behavior of the density for different values of a is given in the following Figure 6.

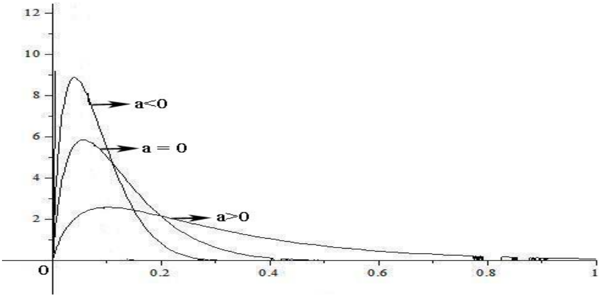

Figure 6: Gamma appended with Bessel function

Pathway idea

- Consider a model which can switch around to three functional forms covering almost all statistical densities in current use, see Mathai(2005). Let

$$f_1^*(x) = c_1^*|x|^\gamma[1-a(1-\alpha)|x|^\delta)^{\frac{\eta}{1-\alpha}}, \alpha < 1, \eta > 0, a > 0, \delta > 0 \qquad (3.1)$$

and $1 - a(1-\alpha|x|^\delta > 0$, where c_1^* is the normalizing constant.

- When $\alpha < 1$ the model in (3.1) stays as the generalized type-1 beta family, extended over the real line.
- When $\alpha > 1$ write $1-\alpha = -(\alpha - 1)$ with $\alpha > 1$. Then the functional form in (3.1) changes to

$$f_2^*(x) = c_2^*|x|^\gamma[1+a(\alpha-1)|x|^\delta]^{-\frac{\eta}{\alpha-1}} \qquad (3.2)$$

for $\alpha > 1, a > 0, \eta > 0, -\infty < x < \infty$. Note that (3.2) is the extended generalized type-2 beta family of functions.

- When $\alpha \to 1$ then both (3.1) and (3.2) go to

$$f_3^*(x) = c_3^*\ |x|^\gamma e^{-a\eta|x|^\delta}, a > 0, \eta > 0, \delta > 0, -\infty < x < \infty. \qquad (3.3)$$

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- Thus (3.1) is capable of switching around to three families of functions. This is the pathway idea and α here is the pathway parameter. Through this parameter α one can reach the three families of functions in (3.1),(3.2),(3.3). Pathway idea was introduced by Mathai (2005).
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Reaction rate probability integral model

- The basic model is an integral of the following form:

$$I_{(1)} = \int_0^\infty x^{\gamma-1} e^{-ax^\delta - zx^{-\rho}}, a > 0, z > 0, \rho > 0, \delta > 0. \qquad (4.1)$$

- For $\rho = \frac{1}{2}, \delta = 1$ one has the basic probability integral in the non-resonant case, see Haubold and Mathai (1988).
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- By applying the Mellin convolution technique we get

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$$= \frac{1}{\rho\delta a^{\frac{\gamma}{\delta}}} H^{2,0}_{0,2}\left[z^{\frac{1}{\rho}} a^{\frac{1}{\delta}}\Big|_{(0,\frac{1}{\rho}),(\frac{\gamma}{\delta},\frac{1}{\delta})}\right] \qquad (4.4)$$

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Generalization of reaction-rate models

- Consider the integral

$$I_{(2)} = \int_0^\infty x^\gamma \mathrm{e}^{-ax^\delta - zx^\rho} \mathrm{d}x, a > 0, \delta > 0, \rho > 0, z > 0. \quad (4.5)$$

- In (4.1) we had $x^{-\rho}$ with $\rho > 0$ whereas in (4.5) we have x^ρ with $\rho > 0$.
- For $\delta = 1$, (4.5) corresponds to the Laplace transform or moment generating function of a generalized gamma density in statistical distribution theory.
- (4.5) can be written in the form of an integral of the form

$$\int_0^\infty v f_1(v) f_2(uv) \mathrm{d}v, f_1(x) = x^{\gamma-1} \mathrm{e}^{-ax^\delta}, f_2(y) = \mathrm{e}^{-y^\rho} \quad (4.6)$$

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- For $\delta = 1$, (4.5) corresponds to the Laplace transform or moment generating function of a generalized gamma density in statistical distribution theory.
- (4.5) can be written in the form of an integral of the form

$$\int_0^\infty v f_1(v) f_2(uv) \mathrm{d}v, f_1(x) = x^{\gamma-1}\mathrm{e}^{-ax^\delta}, f_2(y) = \mathrm{e}^{-y^\rho} \qquad (4.6)$$

for $u = z^{\frac{1}{\rho}}$.

- The integral in (4.6) is in the structure of a Mellin transform of a ratio $u = \frac{y}{x}$ so that by applying Mellin convolution technique we can evaluate I_2.
- A generalization of $I_{(1)}$ and $I_{(2)}$ is the pathway generalized model, which results in the versatile integral.

Generalization of reaction-rate models

- Consider the integral

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Generalization of reaction-rate models

- Consider the integrals of the following types:

$$I_p = \int_0^\infty x^\gamma [1 + a(q_1 - 1)x^\delta]^{-\frac{1}{q_1 - 1}} [1 + b(q_2 - 1)x^\rho]^{-\frac{1}{q_2 - 1}} \mathrm{d}x \qquad (4.8)$$

where $q_1 > 1, q_2 > 1, a > 0, b > 0$,. We will keep ρ free, could be negative or positive.

- Here

$$\lim_{q_1 \to 1, q_2 \to 1} I_p = \int_0^\infty x^\gamma \mathrm{e}^{-ax^\delta - bx^\rho} \mathrm{d}x$$

which is the integral in (4.5) and if $\rho < 0$ then it is the integral in (4.1).

- The general integral in (4.8) belongs to the general family of versatile integrals.
- The whole collection of such models is known as the versatile integrals applicable in a wide variety of situations.
- Integral transforms, known as *P*-transforms, are also associated with the integrals in (4.8), see for example Kumar and Kilbas (2010), Kumar (2011).

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Fractional calculus models

- Fractional integral operators of the second kind or right-sided fractional integral operators can be considered as Mellin convolution of a product as in (4.2) and left-sided or fractional integral operators of the first kind can be considered as Mellin convolution of a ratio where the functions f_1 and f_2 are of the following forms:

$$f_1(x) = \phi_1(x)(1-x)^{\alpha-1}, 0 \le x \le 1, f_2(y) = \phi_2(y)f(y) \tag{4.9}$$

  where ϕ_1 and ϕ_2 are pre-fixed functions, $f(y)$ is arbitrary and $f_1(x) = 0$ outside the interval $0 \le x \le 1$.
- Thus, essentially, all fractional integral operators belong to the categories of Mellin convolution of a product or ratio where one function is a multiple of type-1 beta form and the other is arbitrary.
- The right-sided or type-2 fractional integral of order α is denoted by $D_{2,u}^{-\alpha} f$ and defined as

$$D_{2,u}^{-\alpha} f = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v \tag{4.10}$$

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Fractional calculus models

- Let $D = \frac{\mathrm{d}}{\mathrm{d}u}$ the ordinary derivative with respect to u and D^n be the n-th order derivative. Then the fractional derivative of order α is defined as

$$D^{\alpha} f = D^n [D_{i,u}^{-(n-\alpha)} f] \text{ in the Riemann-Liouville sense and}$$

$$D^{\alpha} f = [D_{i,u}^{-(n-\alpha)} f] D^n \text{ in the Caputo sense} \tag{4.12}$$

  for $i = 1, 2,$ where n be a positive integer such that $\Re(n - \alpha) > 0$.

- The input-output model when applied to reaction-diffusion problems can result in fractional order reaction-diffusion differential equations.
- Such fractional order differential equations are seen to provide solutions which are more relevant to practical situations compared to the solutions coming from differential equations in the conventional sense or involving integer-order derivatives. For more details see, Haubold and Mathai (2000, 2002), Haubold et al. (2010, 2011).

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Analytic solar models

- Suppose the matter density distribution has the following form:

$$f(x) = \rho_0[1 - x^\delta]^\gamma, 0 \leq x \leq 1 \quad (5.1)$$

where $x = \frac{r}{r_S}$ where r_S is the radius of the Sun and r is an arbitrary distance from the center of the Sun.

- Here ρ_0 is the core density or the constant, when $r \to 0$, δ and γ are parameters to be adjusted so that mass, pressure, luminosity etc will match with observational data.
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Acknowledgement

The authors would like to thank the Department of Science and Technology, Government of India, New Delhi, for the financial assistance for this work under Project Number SR/S4/MS:287/05, and the Centre for Mathematical Sciences for providing all facilities.

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